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Orbital Planar Maneuvers Using Two and Three-Four (Through Infinity) Impulses

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We consider the problem of minimum ΔV time-free impulsive transfers between coplanar Keplerian orbits. We studied two types of maneuvers: the ones performed with two impulses and the ones performed with three or four impulses that go to infinity in the middle of the transfer. For the two-impulse maneuver, we develop optimality conditions that lead to a nonlinear system of three equations and three unknowns. For the three-impulse maneuver, we develop a new maneuver that uses two elliptic transfer orbits that are connected by a negligible impulse applied at an infinite distance from the attracting body. It is an extension of the bi-elliptic transfer, where the two orbits involved in the transfer are not coaxial. We study in detail and show regions of optimality for the most trivial cases of transfers: between two circular orbits, one circular and one elliptic orbit, and two elliptic coaxial orbits. We complete the research by studying a scheme to reduce the total ΔV for some of those maneuvers, by adding a second impulse at infinity, and making it a four-impulse maneuver.

Introduction

THIS paper studies the problem of time-free transfers between two elliptical coplanar orbits that extremize the ΔV (fuel consumed). The problem of optimal transfers (in the sense of reducing the fuel consumption) between two Keplerian coplanar orbits has been under investigation for more than 40 years. In particular, many papers solve this problem for an impulsive thrust system with a fixed number of impulses. The literature presents many solutions for particular cases, such as the Hohmann[1] and the Hoelker and Silber[2] transfers between two circular orbits and their variants for ellipses in particular geometry.

In this paper, the equations that give the solution of this problem for a transfer between two elliptic coplanar orbits with two impulses and three or four impulses (through infinity) are derived. For the two-impulse transfer case, we followed the idea developed by Lawden.[3,4] The new aspect of the formulation presented is the introduction of a new set of variables that allows the reduction of his 11 equations in 11 unknowns to a set of 3 equations in 3 unknowns. For the case of three and four impulses, other schemes are developed to solve the problem, considering only transfers that go through infinity during the transfer. We also present numerical tests for the equations derived, showing the savings obtained by the application of more than two impulses.

Review of the Literature

Goddard[5] was one of the first researchers to work on the problem of optimal transfers of a spacecraft between two points. He proposed optimal approximate solutions for the problem of sending a rocket to high altitudes with minimum fuel consumption.

After him came the very important work done by Hohmann,[1] who solved the problem of minimum ΔV transfers between two circular coplanar orbits. His result is largely used nowadays, as a first approximation of more complex models. Later, Hoelker and Silber[2] (and others) showed that this transfer was not the best in all

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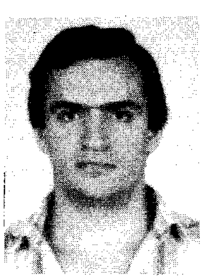

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cases. A detailed study of this transfer can be found in Marec[6] and an analytical proof of its optimality in Barrar.[7]

Next, the Hohmann transfer was generalized to the elliptic case (transfer between two coaxial elliptic orbits) by Marchal.[8] Smith[9] shows results for some other special cases, like coaxial and quasi-coaxial elliptic orbits, circular-elliptic orbits, and two quasicircular orbits. A numerical scheme to solve the transfer between two generic coplanar elliptic orbits is presented by Bender.[10]

The three-impulse concept is introduced in the literature by Shternfeld[11] in Russia. He derived the bi-elliptic transfer (according to Edelbaum[12]). This transfer was later independently derived by Hoelker and Silber[2] and Edelbaum.[13] All of those researchers show that it is possible to find a bi-elliptical transfer between two circular orbits that has a ΔV lower than the one for the Hohmann transfer, when the ratio between the radius of the initial and the final orbits is greater than 11.93875. Later, Roth[14] obtains the minimum ΔV solution for a bi-elliptical transfer between two inclined orbits.

Following the idea of more than two impulses, there are also the papers by Prussing,[15] which admits two or three impulses; Prussing,[16] which admits four impulses, and Eckel,[17] which admits N impulses.

Another line of research that comes from the Hohmann transfer is the study of multirevolutions transfers, with N impulses applied during N successive passages by the apses. Spencer et al.[18] show equations and graphs to obtain the ΔV required for those transfers as a function of the number of revolutions allowed for the maneuver. Next, Redding,[19] Matogawa,[20] and Melton et al.[21] extend this concept of multirevolution transfer to the nonimpulsive case, by applying finite thrust around the apses.

Some other researchers worked on methods where the number of impulses is a free parameter and not a value fixed in advance. It is the case of the papers made by Lion and Handelsman[22] and Jezewski and Rozendaal.[23] Most of the research done in this particular case is based on the primer-vector theory developed by Lawden.[24,25]

Two papers that document and summarize the knowledge about impulsive transfers are the ones written by Edelbaum[12] and Gobetz and Doll.[26]

Bi-Impulsive Transfer

Suppose that a spacecraft is in a Keplerian orbit O_0. It is desired to transfer this spacecraft to a final Keplerian orbit O_2, coplanar with O_0. Figure 1 shows a sketch of the transfer. At point P_1 (distance from the attracting body $= r_1$, angle from the horizontal axis $= \theta_1$), we apply an impulse with magnitude ΔV_1 that has an angle ϕ_1 with the local transverse direction. The transfer orbit crosses the final orbit at point P_2 (distance from the attracting body $= r_2$, angle from the horizontal axis $= \theta_2$), where we apply an impulse with magnitude ΔV_2 making an angle ϕ_2 with the local transverse direction. Remember that P_1 and P_2 are two generic points that belong to the initial and final orbits, respectively. They are not fixed points, and θ_1 and θ_2 are free variables.

Using basic equations from the two-body celestial mechanics, it is possible to write an analytical expression for the total ΔV

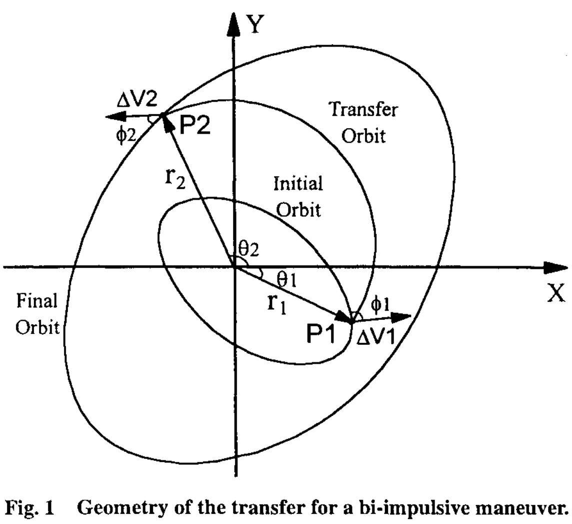

Fig. 1 Geometry of the transfer for a bi-impulsive maneuver.

($=\Delta V_1 + \Delta V_2$) required for this maneuver. To specify each of the three orbits involved in the problem, the elements D, h, and k are used. They are defined by the following equations:

$$D = \mu/C; \qquad k = e\cos(\omega); \qquad h = e\sin(\omega) \tag{1}$$

where μ is the gravitational parameter of the central body, C ($= |\boldsymbol{r} \times \boldsymbol{v}|$) is the angular momentum of the orbit, e is the eccentricity, and ω is the argument of the periapse. The subscripts 0 for the initial orbit, 1 for the transfer orbit, and 2 for the final orbit are also used. In those variables, the expressions for the radial (subscript r) and transverse (subscript t) components of the two impulses, as a function of the five variables $D_1, \theta_1, \theta_2, h_1, k_1$, are

$$\Delta V_{r1} = (D_1k_1 - D_0k_0)\sin(\theta_1) - (D_1h_1 - D_0h_0)\cos(\theta_1) \tag{2}$$

$$\Delta V_{t1} = D_1 - D_0 + (D_1k_1 - D_0k_0)\cos(\theta_1) + (D_1h_1 - D_0h_0)\sin(\theta_1) \tag{3}$$

$$\Delta V_{r2} = (D_2k_2 - D_1k_1)\sin(\theta_2) - (D_2h_2 - D_1h_1)\cos(\theta_2) \tag{4}$$

$$\Delta V_{t2} = D_2 - D_1 + (D_2k_2 - D_1k_1)\cos(\theta_2) + (D_2h_2 - D_1h_1)\sin(\theta_2) \tag{5}$$

Equations (2–5) are obtained from the equations for the radial velocity

$$V_r = \frac{e\sin(\theta_1 - \omega_0)}{\sqrt{a(1-e^2)}}$$

and transverse velocity

$$V_t = \frac{1 + e\cos(\theta_1 - \omega_0)}{\sqrt{a(1-e^2)}}$$

where $\theta_1 - \omega_0$ is the true anomaly of the spacecraft. As an illustration, we will derive Eq. (2). The radial velocity of the spacecraft, when in its initial orbit, gives us the following result:

$$V_{r0} = e_0\sin(\theta_1 - \omega_0)\Big/\sqrt{a_0\left(1 - e_0^2\right)}$$
$$= (e_0\sin\theta_1\cos\omega_0 - e_0\cos\theta_1\sin\omega_0)/\sqrt{p_0}$$

{using the facts that $\mu = 1$ and $p_0 = \sqrt{[a_0(1 - e_0^2)]}\} = D_0(k_0\sin\theta_1 - h_0\cos\theta_1)$ [using Eqs. (1) $C = \sqrt{p}$]. Then, we can calculate ΔV_{r1} as the difference between the radial velocities of the spacecraft at θ_1 when it is in the initial and when it is in the transfer orbit. Equations (3–5) are obtained using the same procedure.

The problem now is to find the transfer orbit that minimizes the total ΔV and satisfies the following two constraint equations, expressing the fact that the transfer orbit intersects the initial and final orbits:

$$g_1 = D_0^2[1 + k_0\cos(\theta_1) + h_0\sin(\theta_1)] - D_1^2[1 + k_1\cos(\theta_1) + h_1\sin(\theta_1)] = 0 \tag{6}$$

$$g_2 = D_2^2[1 + k_2\cos(\theta_2) + h_2\sin(\theta_2)] - D_1^2[1 + k_1\cos(\theta_2) + h_1\sin(\theta_2)] = 0 \tag{7}$$

Equation (6) is obtained by calculating the distance from the spacecraft to the attracting body at θ_1, when the spacecraft is in the initial orbit (r_0), and when it is in the transfer orbit (r_1) (they have to be the same, because the orbits intercept). The distance r_0 is given by

$$r_0 = p_0/[1 + e_0\cos(\theta_1 - \omega_0)]$$
$$= p_0/(1 + e_0\cos\theta_1\cos\omega_0 + e_0\sin\theta_1\sin\omega_0)$$
$$= 1/D_0^2(1 + k_0\cos\theta_1 + h_0\sin\theta_1)$$

Performing the same calculations for r_1 and using the fact that $r_0 = r_1$, we can obtain Eq. (6). Equation (7) is obtained using the same procedure at θ_2, for the transfer and final orbits.

In mathematical language, our problem is to minimize $\Delta V = \sqrt{(\Delta V_{r1}^2 + \Delta V_{t1}^2)} + \sqrt{(\Delta V_{r2}^2 + \Delta V_{t2}^2)}$ subject to the conditions (6) and (7). The constraints (6) and (7) can be used to solve this system for two of the variables, making the equation for the ΔV a function of only three independent variables. After algebraic manipulations, one can find the following equations:

$$k_1 = -\csc(\theta_1 - \theta_2) \times \left(\left\{\left(D_0^2/D_1^2\right)[1 + k_0\cos(\theta_1) + h_0\sin(\theta_1)] - 1\right\}\sin(\theta_2) \times \left\{\left(D_2^2/D_1^2\right)[1 + k_2\cos(\theta_2) + h_2\sin(\theta_2)] - 1\right\}\sin(\theta_1)\right) \tag{8}$$

$$h_1 = -\csc(\theta_1 - \theta_2) \times \left(\left\{\left(D_2^2/D_1^2\right)[1 + k_2\cos(\theta_2) + h_2\sin(\theta_2)] - 1\right\}\cos(\theta_1) - \left\{\left(D_0^2/D_1^2\right)[1 + k_0\cos(\theta_1) + h_0\sin(\theta_1)] - 1\right\}\cos(\theta_2)\right) \tag{9}$$

Now that ΔV is a function of only three variables (D_1, θ_1, and θ_2), elementary calculus can be used to find its minimum. From the definition of ΔV it is possible to write

$$\frac{\partial(\Delta V)}{\partial\alpha_m} = 0 = \frac{1}{\Delta V_1}\left[\Delta V_{r1}\frac{\partial(\Delta V_{r1})}{\partial\alpha_m} + \Delta V_{t1}\frac{\partial(\Delta V_{t1})}{\partial\alpha_m}\right] + \frac{1}{\Delta V_2}\left[\Delta V_{r2}\frac{\partial(\Delta V_{r2})}{\partial\alpha_m} + \Delta V_{t2}\frac{\partial(\Delta V_{t2})}{\partial\alpha_m}\right] \tag{10}$$

where $m = 1, 2, 3$ and $\alpha_1 = D_1$, $\alpha_2 = \theta_1$, and $\alpha_3 = \theta_2$.

The chain rule for derivatives can be applied to obtain expressions for the partials involved in Eq. (10). A general expression for them is

$$\frac{\partial(\Delta V_{ij})}{\partial\alpha_m} = \left.\frac{\partial(\Delta V_{ij})}{\partial\alpha_m}\right|_{\text{Direct}} + \frac{\partial(\Delta V_{ij})}{\partial k_1}\frac{\partial k_1}{\partial\alpha_m} + \frac{\partial(\Delta V_{ij})}{\partial h_1}\frac{\partial h_1}{\partial\alpha_m} \tag{11}$$

where $i = r, t$; $j = 1, 2$; and Direct stands for the part of the derivative that comes from the explicit dependence of ΔV_{ij} in the variable α_m. The expressions for $[\partial(\Delta V_{ij})/\partial k_1]$ and $[\partial(\Delta V_{ij})/\partial h_1]$ can be obtained from Eqs. (2–5), and the expressions for $(\partial k_1/\partial\alpha_m)$ and $(\partial h_1/\partial\alpha_m)$ can be obtained from the Eqs. (8) and (9). Equations (10) have multiple solutions. The way we solve those equations is to perform a previous mapping of the values of the ΔV by varying D_1, θ_1, and θ_2, in specified ranges. This mapping can provide a good first guess to solve Eqs. (10) with an iterative process and to find the global minimum of the ΔV.

180-deg Transfer Theory

It is easy to see the existence of singularities in Eqs. (8) and (9) that happen for the transfers where $\theta_1 - \theta_2 = m\pi$, where m is any integer. This particular case includes the very important Hohmann-class family. It occurs when the transfer is between two circular orbits, between a circular and an elliptic orbit, or between two elliptic coaxial orbits (ellipses with the semimajor axis in the same direction). From the initial data (initial and final orbits) it is known in advance if the problem is one of those cases, and it can be solved by trivial means (such as the Hohmann transfer), without going through this theory. Alternatively, if desired, this theory can be adapted to solve this class of transfers. The procedure is to solve the constraints (6) and (7) for the variables D_1 and k_1 and to follow the same guidelines after that. The new independent variables are now h_1, θ_1, and θ_2. A full development of this 180-deg transfer theory is not shown here, because Eqs. (8) and (9) generate less algebraic work in the continuation. This is because Eqs. (6) and (7) are symmetric in the variables h_1 and k_1. The expressions for D_1 and k_1 are the only development shown here. They are

$$D_1 = \sqrt{\frac{D_2^2\cos(\theta_1)[1 + h_2\sin(\theta_2)] - D_0^2\cos(\theta_2)[1 + h_0\sin(\theta_1)] + \left[D_2^2k_2 - D_0^2k_0\right][\cos(\theta_1)\cos(\theta_2)]}{\cos(\theta_1) - \cos(\theta_2) - h_1\sin(\theta_1 - \theta_2)}} \tag{12}$$

$$k_1 = \frac{D_0^2 - D_2^2 + D_0^2k_0\cos(\theta_1)[1 + h_1\sin(\theta_2)] - D_2^2k_2\cos(\theta_2)[1 + h_1\sin(\theta_1)] + \left[D_0^2h_0 - D_2^2h_1\right]\sin(\theta_1)}{D_2^2\cos(\theta_1)[1 + h_2\sin(\theta_2)] - D_0^2\cos(\theta_2)[1 + h_0\sin(\theta_1)] - \left[D_0^2k_0 - D_2^2k_2\right]\cos(\theta_1)\cos(\theta_2)} + \frac{\left[D_0^2h_1 - D_2^2h_2\right]\sin(\theta_2) + \left[D_0^2h_0 - D_2^2h_2\right]h_1\sin(\theta_2)\sin(\theta_1)}{D_2^2\cos(\theta_1)[1 + h_2\sin(\theta_2)] - D_0^2\cos(\theta_2)[1 + h_0\sin(\theta_1)] - \left[D_0^2k_0 - D_2^2k_2\right]\cos(\theta_1)\cos(\theta_2)} \tag{13}$$

It is possible to see that those equations have no singularities for $\theta_1 - \theta_2 = m\pi$. Singularities in other points are not important, because this variant of the theory is used only for the cases where $\theta_1 - \theta_2 = m\pi$. The cases where $\theta_1 - \theta_2 \neq m\pi$ are covered by the equations derived before. The validity of this theoretical variant can be checked by verifying that the Eqs. (12) and (13) give the right solutions for the trivial cases cited before (circular-circular, circular-ellipse, ellipse-ellipse coaxial).

Three-Impulse Transfers Through Infinity

For the trivial cases cited before, different possibilities of three-impulse transfers are compared with each other and with the two-impulse maneuver. For the more generic case of two elliptic and noncoaxial orbits, a new approach and a new set of equations are developed to solve this problem. They are shown in detail, case by case, in the following sections.

Transfers Between Two Circular Orbits

After the discovery of the Hohmann transfer between two coplanar circular orbits,[1] many other important steps were made by several researchers. Shternfeld,[11] Edelbaum,[13] and Hoelker and Silber[2] showed that it is possible to find a bi-elliptical three-impulsive transfer that has a lower ΔV, when the ratio of the radius of the two orbits involved is greater than 11.93875. This transfer is accomplished in three steps: 1) the first impulse is applied to send the spacecraft from its initial orbit to a first elliptic transfer orbit with apoapse at a distance R_A (greater than the radius of the final orbit); 2) at this point, a second impulse is applied, with near-zero ΔV, to transfer the spacecraft to a second elliptic transfer orbit that will put the spacecraft in a path that crosses the final orbit; 3) the last impulse is applied when the spacecraft crosses its final orbit, and this impulse makes the spacecraft stay in that orbit. The savings provided by this transfer increase when the distance R_A of the point where the intermediate impulse is applied is increased, at an expense of an increase in the time for the transfer. The maximum for the savings in ΔV is given by the biparabolic transfer, where both transfer orbits are parabolic and the second impulse is applied at an infinite distance with a zero magnitude. In this section, the variables that were defined previously are used again to obtain the equations for the savings that a transfer through infinity can give over the standard bi-impulse maneuver. The expression for the extra time required to complete the transfer is also obtained. The savings ΔV_{SAV} defined as the difference in ΔV between the three-impulsive transfer (ΔV_{TRI}) and the standard Hohmann transfer (ΔV_{HOH}) is given by

$$\Delta V_{SAV} = \Delta V_{TRI} - \Delta V_{HOH} \tag{14}$$

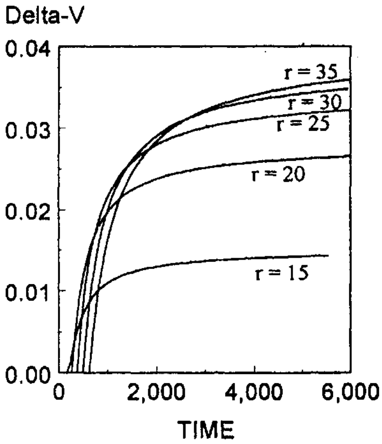

Fig. 2 ΔV saved by using three impulses vs $\Delta T_{\mathrm{EXTRA}}$.

where

$$\Delta V_{\mathrm{TRI}} = D_0\left[-1 - \frac{1}{r} + \sqrt{2\left(\frac{1}{r^2} + \frac{1}{D_0^2 R_A}\right)} + \frac{\sqrt{2}(D_0^2 R_A - 1)}{D_0\sqrt{R_A(1 + D_0^2 R_A)}}\right] \tag{15}$$

$$\Delta V_{\mathrm{HOH}} = D_0\left[-1 + \frac{1}{r} + \frac{\sqrt{2}(r^2 - 1)}{r\sqrt{1 + r^2}}\right] \tag{16}$$

$$r = D_0/D_2 = \sqrt{a_2/a_0} \tag{17}$$

where R_A is the distance between the attracting body and the spacecraft at the moment when the second impulse is applied, and a_0 and a_2 are the semimajor axes of the initial and the final orbits, respectively.

If only the limit case $R_A \to \infty$ is considered, the expression for ΔV_{SAV} goes to

$$\Delta V_{\mathrm{SAV}}(R_A \to \infty) = D_0\left[\sqrt{2} - \frac{2}{r} + \frac{\sqrt{2}}{r} + \frac{\sqrt{2}(1 - r^2)}{r\sqrt{1 + r^2}}\right] \tag{18}$$

The extra time required for these transfers ($\Delta T_{\mathrm{EXTRA}}$), which means T_{TRI} (time for the three-impulse transfer) $-$ T_{HOH} (time for the Hohmann transfer), is given by

$$\Delta T_{\mathrm{EXTRA}} = T_{\mathrm{TRI}} - T_{\mathrm{HOH}} \tag{19}$$

where

$$T_{\mathrm{TRI}} = \pi\left[\left(\frac{r_2 + D_0^2 R_A}{2D_0^2}\right)^{\frac{3}{2}} + \left(\frac{1 + D_0^2 R_A}{2D_0^2}\right)^{\frac{3}{2}}\right] \tag{20}$$

$$T_{\mathrm{HOH}} = \pi\left(\frac{1 + r^2}{2D_0^2}\right)^{\frac{3}{2}} \tag{21}$$

It is known that not all of the cases (combination of initial and final orbit) can provide savings by using a three-impulse maneuver. This possibility depends on the value of r. The smallest value for r that allows a positive saving is given by

$$\lim_{R_A \to \infty} \Delta V_{\mathrm{SAV}} = D_0\left[\sqrt{2} - \frac{2}{r} + \frac{\sqrt{2}}{r} + \frac{\sqrt{2}(1 - r^2)}{r\sqrt{1 + r^2}}\right] = 0 \Rightarrow r = 3.45525 \Rightarrow r^2 = \frac{a_2}{a_0} = 11.93875 \tag{22}$$

This result is in agreement with the literature.[6] When this ratio (r^2) is greater than 15.582, any bi-elliptical transfer is better than the Hohmann transfer.

A plot of ΔV_{SAV} vs $\Delta T_{\mathrm{EXTRA}}$ is shown in Fig. 2 for several values of r. The units are chosen such that the gravitational parameter and the quantity D_0 are both one, which corresponds to the canonical system of units for this case.

It is easy to see that the savings increase when more time is allowed for the transfer and that all of the savings go to a certain limit (the biparabolic transfer) when the time is large enough.

Transfers Between One Circular and One Elliptic Orbit

This case is a little more complex, and it provides two choices for the Hohmann-type transfers and two choices for the three-impulse transfers. In both cases there is a transfer orbit that inserts the spacecraft at the apoapse (H1 and TRI2) or at the periapse (H2 and TRI1) of the final orbit.

Marchal[8] shows that between the two choices available for the Hohmann-type transfer, the one with lower ΔV is the one that uses the apoapse of the elliptic orbit. This is the transfer H1, according to the nomenclature previously defined. It is easy to see that this H1 requires less ΔV, but it requires more time for the transfer because the semimajor axis of the transfer orbit is larger.

To find out which one of the three-impulse transfers has a lower ΔV, the differences in ΔV for both cases, in the limiting situation where the second ΔV is applied at an infinite distance, are calculated. The final result, after algebraic manipulations, is

$$\Delta V|_{\mathrm{TRI1-TRI2}} = \lim_{R_A \to \infty}(\Delta V_{\mathrm{TRI1}} - \Delta V_{\mathrm{TRI2}}) = D_2[k_2 + \sqrt{2(1 - k_2)} - \sqrt{2(1 + k_2)}] \tag{23}$$

If the fact that $D_2 > 0$ is considered, it is possible to study the expression (23) to get the following conclusions: $\Delta V|_{\mathrm{TRI1-TRI2}} > 0$, if $k_2 > 0$ and $\Delta V|_{\mathrm{TRI1-TRI2}} < 0$, if $k_2 < 0$. Those conclusions can be expressed in words by saying that the three-impulsive transfer that inserts the spacecraft at the periapse of the second orbit has a smaller ΔV. Another advantage of TRI1 is that it requires a smaller time for the transfer (for a fixed value of R_A).

The next step is to compare the transfers with two and three impulses with each other. This is done by deriving the equations for the savings in ΔV and for the extra time required for the transfer, in the same way it was done in the previous section. The final results are

$$\Delta V_{\mathrm{SAV}} = \Delta V_{\mathrm{TRI}} - \Delta V_{\mathrm{HOH}} \tag{24}$$

where

$$\Delta V_{\mathrm{HOH}} = D_2 - D_0 - D_2 k_2 + \frac{\sqrt{2}\left[D_0^2 - D_2^2(1 - k_2)\right]}{\sqrt{D_0^2 + D_2^2(1 - k_2)}} \tag{25}$$

is the ΔV for the Hohmann-type transfer and

$$\Delta V_{\mathrm{TRI}} = -D_0 - D_2 - D_2 k_2 + \sqrt{2\left[D_2^2(1 + k_2) + (1/R_A)\right]} \tag{26}$$

is the ΔV for the three-impulse transfer that requires a smaller impulse, between the two choices compared. For the extra time required for the transfer, the results are

$$\Delta T_{\mathrm{EXTRA}} = T_{\mathrm{TRI}} - T_{\mathrm{HOH}} \tag{27}$$

where

$$T_{\mathrm{TRI}} = \pi\left\{\sqrt{\left(\frac{1}{2D_0^2} + \frac{R_A}{2}\right)^3} + \sqrt{\left[\frac{1}{2D_2^2(1 + k_2)} + \frac{R_A}{2}\right]^3}\right\} \tag{28}$$

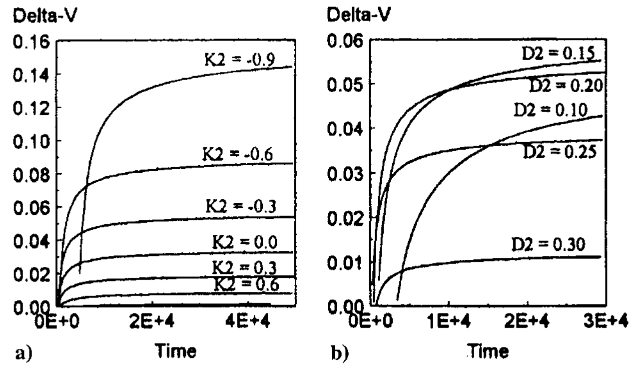

Fig. 3 ΔV **saved by using three impulses vs time for the transfer: a)** $D_0 = 1, D_2 = 0.2, -0.9 \leq k_1 \leq 0.9$ **and b)** $D_0 = 1, k_2 = -0.3, 0.1 \leq D_2 \leq 0.3$.

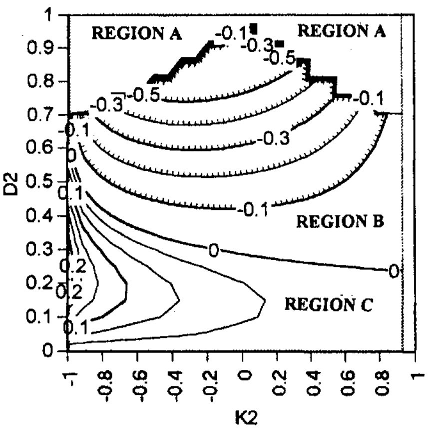

Fig. 4 Level curves for ΔV_{SAV}.

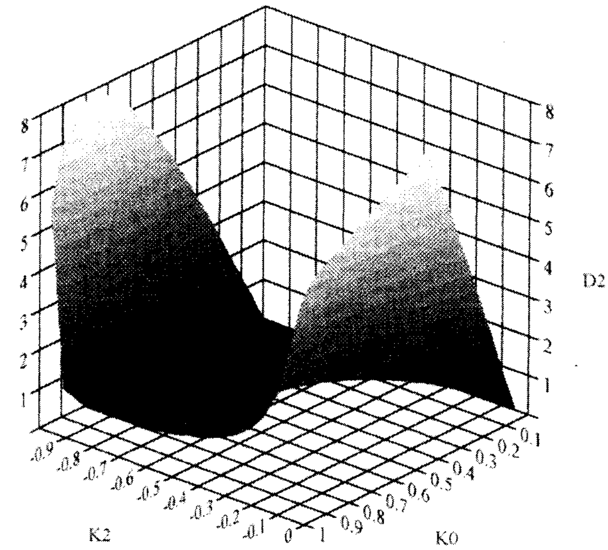

Fig. 5 Surface $\Delta V|_{T1-T2} = 0$ **in the volume** (k_0, k_2, **and** D_2).

is the time required to complete the three-impulse transfer and

$$T_{\text{HOH}} = \pi\sqrt{\left[\frac{1}{2D_0^2} + \frac{1}{2D_2^2(1-k_2)}\right]^3} \tag{29}$$

is the time required to complete the Hohmann transfer.

Figure 3 shows ΔV_{SAV} vs ΔT_{EXTRA} for the case $D_0 = 1$ as a function of D_2 and k_2. Figure 4 shows the level curves for ΔV_{SAV}, to show the region of the two-dimensional space $D_2 - k_2$ that allows savings in ΔV by using a transfer through infinity. It is possible to see the existence of three regions: region A is the forbidden region of the equations derived here, where the orbits intersect and the equations have to be modified; region B is the region where the savings are negative and the two-impulse maneuver has a lower ΔV; region C is the region where the savings are positive and the three-impulse maneuver has a lower ΔV.

Transfers Between Two Coaxial Elliptic Orbits

For this particular case there are several different possibilities for the geometry. First, the orbits may have their periapse in the same direction (aligned orbits) or in opposite direction (opposite orbits). For each of those situations, there are two possibilities for a Hohmann-type orbit and two for a three-impulse type transfer orbit: the transfer can start at the periapse or at the apoapse of the initial orbit. All of those cases are organized and named in the following way: H1, a Hohmann-type transfer starting at the periapse of the initial orbit; H2, a Hohmann-type transfer starting at the apoapse of the initial orbit; TRI1, a three-impulse transfer starting at the periapse of the initial orbit; and TRI2, a three-impulse transfer starting at the apoapse of the initial orbit. Both of the three-impulse transfers go to infinity.

The literature[8] has a rule to choose the transfer that requires a lower ΔV between the two Hohmann-type transfers: it is the one that uses the most distant apoapse.

In this research, the problem of comparing the two three-impulse transfers between themselves and the three-impulse with the Hohmann-type transfers is addressed.

For the comparison of the two three-impulse transfers, an expression for the difference in ΔV between them is derived. For this comparison, we assume the limit case for both transfers, which means that both three-impulse transfers go through infinity in the intermediate step. The final expression, after algebraic manipulation, is

$$\begin{aligned}\Delta V_{T1-T2} &= \Delta V_{\text{TRI1}} - \Delta V_{\text{TRI2}}\\ &= D_2\sqrt{2(1+k_2)} - D_2\sqrt{2(1-k_2)} - 2D_2k_2\\ &\quad + \sqrt{2(1+k_0)} - \sqrt{2(1-k_0)} - 2k_0\end{aligned} \tag{30}$$

The reference system is defined such that $0 \leq k_0 \leq 1$ and $-1 \leq k_2 \leq 1$, which means that the angles are measured from the periapse of the initial orbit.

Note that the elements that are used here to describe the orbits (D, h, and k) are very appropriate for this expression because Eq. (30) is valid for aligned and opposed orbits.

From this expression, the following conclusions can be reached. For $0 \leq k_2 \leq 1$ (aligned orbits), ΔV_{T1-T2} is always negative, which means that the maneuver TRI1 is always better than TRI2. For $-1 \leq k_2 \leq 0$ (opposed orbits), ΔV_{T1-T2} can be positive or negative. Then, the equation $\Delta V_{T1-T2} = 0$ is set and solved for D_2 to find the separation point of optimality between the two orbits. The result is

$$D_{2\text{CRI}} = \frac{\sqrt{2(1+k_0)} - \sqrt{2(1-k_0)} - 2k_0}{\sqrt{2(1-k_2)} - \sqrt{2(1+k_2)} + 2k_2} \tag{31}$$

and the following conclusions can be reached: If $D_2 = D_{2\text{CRI}}$, $\Delta V_{\text{TRI1}} = \Delta V_{\text{TRI2}}$; if $D_2 > D_{2\text{CRI}}$, $\Delta V_{\text{TRI1}} < \Delta V_{\text{TRI2}}$; if $D_2 < D_{2\text{CRI}}$, $\Delta V_{\text{TRI1}} > \Delta V_{\text{TRI2}}$.

Figure 5 shows the surface $\Delta V_{T1-T2} = 0$. This surface divides the three-dimensional space (k_0, k_2, and D_2) into two regions: the one below the surface ($D_2 < D_{2\text{CRI}}$) where TRI2 is better than TRI1 and the one above the surface ($D_2 > D_{2\text{CRI}}$) where TRI1 is better than TRI2.

The same procedure used before can be used now to compare the three-impulse transfer with the lower ΔV with the Hohmann-type transfer with the lower ΔV.

First, it is necessary to define some basic equations. The same nomenclature used in the previous cases is used here. For the various ΔV, the expressions are

$$\Delta V_{\text{SAV}} = \Delta V_{\text{TRI}} - \Delta V_{\text{HOH}} \tag{32}$$

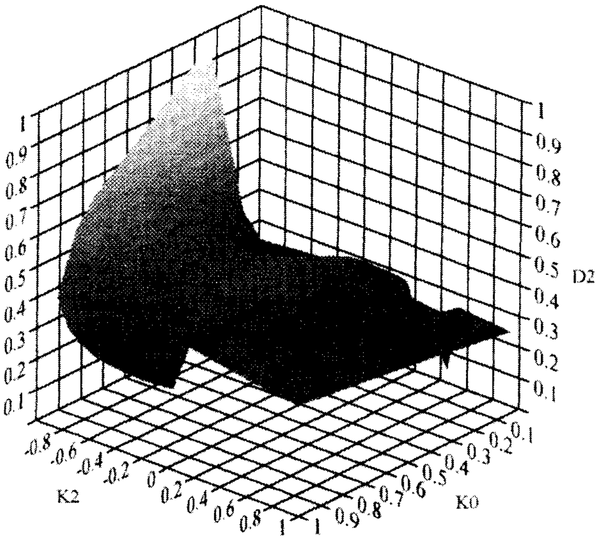

Fig. 6 Surface $\Delta V_{SAV} = 0$ in the volume (k_0, k_2, and D_2).

where

$$\Delta V_{HOH} = D_2 - 1 - k_0 - D_2 k_2 + \sqrt{2\left[1 + D_2^2(1 - k_2) + k_0\right]} \times \left[\frac{2(1 + k_0)}{1 + k_0 + D_2^2(1 - k_2)} - 1\right] \tag{33}$$

for aligned orbits or

$$\Delta V_{HOH} = D_2 - 1 + k_0 + D_2 k_2 - \sqrt{2\left[1 + D_2^2(1 + k_2) - k_0\right]} \times \left[\frac{2(-1 + k_0)}{1 - k_0 + D_2^2(1 + k_2)} + 1\right] \tag{34}$$

for opposed orbits;

$$\Delta V_{TRI1} = -1 - D_2(1 + k_2) - k_0 + \sqrt{2\left(1 + k_0 + \frac{1}{R_A}\right)} \times \left[\frac{(1 + k_0)R_A - 1}{(1 + k_0)R_A + 1}\right] + \sqrt{2\left[\frac{1 + D_2^2 R_A(1 + k_2)}{R_A}\right]} \tag{35}$$

$$\Delta V_{TRI2} = -1 - D_2(1 - k_2) + k_0 - \sqrt{2\left(1 - k_0 + \frac{1}{R_A}\right)} \times \left[\frac{(k_0 - 1)R_A + 1}{(1 - k_0)R_A + 1}\right] + \sqrt{2\left[\frac{1 + D_2^2 R_A(1 - k_2)}{R_A}\right]} \tag{36}$$

Those equations are transformed, in the limit case $R_A, t \to \infty$, in the expressions

$$\Delta V_{TRI1} = \sqrt{2(1 + k_0)} - 1 - D_2\left[1 + k_2 - \sqrt{2(1 + k_2)}\right] - k_0 \tag{37}$$

$$\Delta V_{TRI2} = \sqrt{2(1 - k_0)} - 1 - D_2\left[1 - k_2 - \sqrt{2(1 - k_2)}\right] + k_0 \tag{38}$$

For the extra time required for the transfer, the expressions are

$$\Delta T_{EXTRA} = T_{TRI} - T_{HOH} \tag{39}$$

where for aligned orbits

$$T_{HOH} = \pi\left[\frac{1}{2(1 + k_0)} + \frac{1}{2D_2^2(1 - k_2)}\right]^{\frac{3}{2}} \tag{40}$$

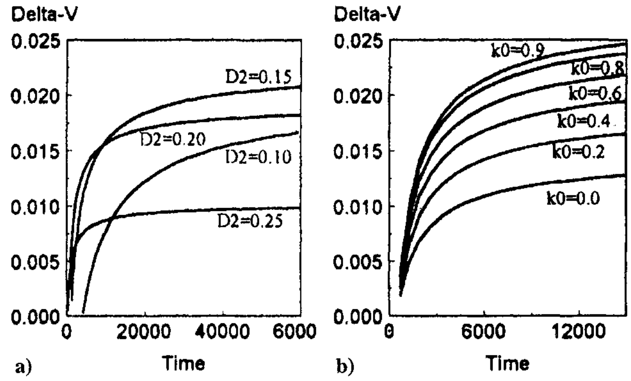

Fig. 7 ΔV saved by using three impulses vs time for the transfer: a) $D_0 = 1, k_0 = 0.2; k_2 = 0.4; 0.10 \leq D_2 \leq 0.25$ and b) $D_0 = 1, k_2 = 0.4; D_2 = 0.2; 0.0 \leq k_0 \leq 0.9$.

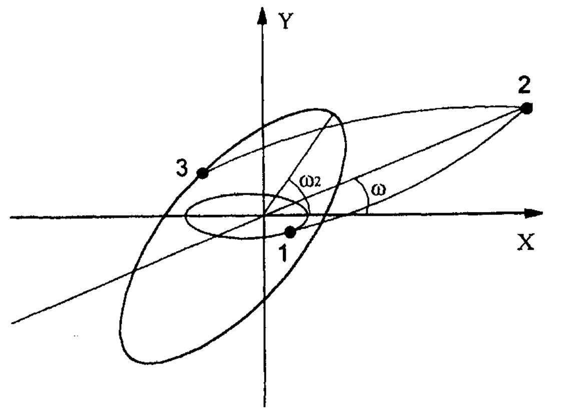

Fig. 8 Geometry of the transfer for a three-impulsive maneuver.

and for opposed orbits

$$T_{HOH} = \pi\left[\frac{1}{2(1 - k_0)} + \frac{1}{2D_2^2(1 + k_2)}\right]^{\frac{3}{2}} \tag{41}$$

$$T_{TRI1} = \pi\left\{\left[\frac{1}{2(1 + k_0)} + \frac{R_A}{2}\right]^{\frac{3}{2}} + \left[\frac{1}{2D_2^2(1 + k_2)} + \frac{R_A}{2}\right]^{\frac{3}{2}}\right\} \tag{42}$$

$$T_{TRI2} = \pi\left\{\left[\frac{1}{2(1 - k_0)} + \frac{R_A}{2}\right]^{\frac{3}{2}} + \left[\frac{1}{2D_2^2(1 - k_2)} + \frac{R_A}{2}\right]^{\frac{3}{2}}\right\} \tag{43}$$

The process of defining a surface $\Delta V_{SAV} = 0$ in the three-dimensional space (k_0, k_2, and D_2) to divide the volume into two parts is repeated here: the region below the surface is the region where the three-impulse maneuver is better and the region above the surface is the region where the Hohmann-type maneuver is better. Figure 6 shows this division, and Fig. 7 shows ΔV_{SAV} vs ΔT_{EXTRA} keeping two parameters fixed and varying only the third one. Remember that the limit case ($R_A, t \to \infty$) is always assumed for those calculations.

Transfers Between Any Two Coplanar Elliptic Orbits Through Infinity

In this section a set of equations to find a new three-impulse transfer through infinity between two generic coplanar elliptic orbits that extremize the total ΔV consumed is derived. The coordinate system is defined such that the periapse of the initial orbit lies in the positive region of the horizontal axis. The semimajor axis of the initial orbit is selected as the unit of distance. In this way, it is possible to say that the Keplerian elements (a, e, ω) of those orbits are (1, e, 0) for

Table 1 Results for the two- and three-impulse transfers

Orbital elements				Three-impulse solution			Two-impulse transfer			Three-impulse transfer			Savings
e	e_2	a_2	ω_2	θ_0	θ_1	ω	ΔV_1	ΔV_2	ΔV_{tot}	ΔV_1	ΔV_2	ΔV_{tot}	ΔV_{SAV}
0.3	0.3	10	60	18.4	344.9	10.9	0.2866	0.1620	0.4486	0.3302	0.1178	0.4480	0.0006
0.3	0.3	10	120	14.2	344.8	8.4	0.2743	0.2199	0.4942	0.3291	0.1470	0.4761	0.0181
0.3	0.3	30	60	10.8	339.3	6.4	0.3146	0.1194	0.4340	0.3284	0.0693	0.3977	0.0363
0.3	0.3	30	120	7.8	341.9	4.6	0.3101	0.1622	0.4723	0.3280	0.0857	0.4137	0.0586
0.3	0.6	10	60	48.5	337.0	29.8	0.3177	0.0992	0.4169	0.3461	0.0971	0.4432	−0.0263
0.3	0.6	10	120	26.1	329.6	15.6	0.4062	0.0838	0.4900	0.3329	0.1702	0.5031	−0.0131
0.3	0.6	30	60	26.3	315.2	15.7	0.3305	0.0862	0.4167	0.3329	0.0656	0.3985	0.0182
0.3	0.6	30	120	13.2	325.9	7.8	0.3612	0.2643	0.4955	0.3289	0.1012	0.4301	0.0654
0.6	0.3	10	60	8.3	334.7	2.4	0.1991	0.1992	0.3983	0.2367	0.1219	0.3586	0.0397
0.6	0.3	10	120	5.7	339.8	1.7	0.1846	0.2710	0.4556	0.2364	0.1494	0.3858	0.0698
0.6	0.3	30	60	4.8	333.5	1.4	0.2239	0.2644	0.3583	0.2363	0.0707	0.3070	0.0513
0.6	0.3	30	120	3.2	339.3	0.9	0.2189	0.1822	0.4011	0.2362	0.0864	0.3226	0.0785
0.6	0.6	10	60	18.7	303.7	5.5	0.2110	0.1765	0.3875	0.2392	0.1253	0.3645	0.0230
0.6	0.6	10	120	9.2	323.7	2.7	0.1625	0.3438	0.5063	0.2368	0.1781	0.4149	0.0914
0.6	0.6	30	60	10.5	301.3	3.1	0.2254	0.1515	0.3769	0.2370	0.0739	0.3109	0.0660
0.6	0.6	30	120	5.2	323.1	1.5	0.2138	0.2350	0.4488	0.2363	0.1032	0.3395	0.1093

the initial orbit and (a_2, e_2, ω_2) for the final orbit. The Keplerian elements are used instead of the new elements defined earlier in this paper because this choice gives equations in a simpler form. This maneuver uses two intermediate elliptic (quasiparabolic) orbits with eccentricity assumed to be one and semimajor axis infinity. The maneuver has three basic steps.

1) The first impulse is applied when the spacecraft has true anomaly θ_0 in the initial orbit. This impulse makes the spacecraft go to the first intermediate quasiparabolic transfer orbit with argument of periapse ω.

2) From this orbit, when the spacecraft is at an infinite distance from the attracting body (apoapse of the quasiparabolic orbit), the second impulse is applied (with zero magnitude) to transfer the spacecraft to a second quasiparabolic orbit with the same argument of periapse ω. In this orbit the spacecraft crosses the final orbit in a point that makes an angle θ_1 with a reference line (adopted as the line of apses of the initial orbit).

3) At this crossing point, the last impulse is applied, to capture the spacecraft into its final orbit.

Note that by performing those steps we are splitting the change in the argument of periapse into two parts: 1) from zero to ω in the first impulse and 2) from ω to ω_2 in the third impulse. Figure 8 shows a sketch of this transfer.

The task is to find an expression for the total ΔV required for this transfer $(\Delta V_1 + \Delta V_2)$ as a function of the three independent variables θ_0, θ_1, and ω. This is done by combining some basic equations from celestial mechanics in the way shown next.

First, we calculate the distance from the attracting body (r_0), the radial velocity (V_{r0}), and the transverse velocity (V_{t0}) for the spacecraft when in the initial orbit, just before the first impulse. Then we calculate the semilatus rectum (p_1), the radial velocity (V_{rt0}), and the transverse velocity (V_{tt0}) for the spacecraft when in the first transfer orbit, just after the first impulse. The equations are

$$r_0 = \frac{1 - e^2}{1 + e\cos(\theta_0)} \qquad p_1 = r_0[1 + \cos(\theta_0 - \omega)] \tag{44a}$$

$$V_{r0} = \frac{e\sin(\theta_0)}{\sqrt{1 - e^2}} \qquad V_{rt0} = \frac{\sin(\theta_0 - \omega)}{\sqrt{p_1}} \tag{44b}$$

$$V_{t0} = \frac{1 + e\cos(\theta_0)}{\sqrt{1 - e^2}} \qquad V_{tt0} = \frac{1 + \cos(\theta_0 - \omega)}{\sqrt{p_1}} \tag{44c}$$

Then, it is possible to write

$$\Delta V_1 = \sqrt{(V_{rt0} - V_{r0})^2 + (V_{tt0} - V_{t0})^2}$$

This process is repeated for the third impulse (remember that the second impulse has zero magnitude), and we calculate the distance from the attracting body (r_1), the radial velocity (V_{r1}), and the transverse velocity (V_{t1}) for the spacecraft when in the final orbit, just after the third impulse. Then we calculate the semilatus rectum (p_2), the radial velocity (V_{rt1}), and the transverse velocity (V_{tt1}) for the spacecraft when in the second transfer orbit, just before the third impulse. The equations are

$$r_1 = \frac{a_2\left(1 - e_2^2\right)}{1 + e_2\cos(\theta_1 - \omega_2)} \qquad p_2 = r_1[1 + \cos(\theta_1 - \omega)] \tag{45a}$$

$$V_{r1} = \frac{e_2\sin(\theta_1 - \omega_2)}{\sqrt{a_2\left(1 - e_2^2\right)}} \qquad V_{rt1} = \frac{\sin(\theta_1 - \omega)}{\sqrt{p_2}} \tag{45b}$$

$$V_{t1} = \frac{1 + e_2\cos(\theta_1 - \omega_2)}{\sqrt{a_2\left(1 - e_2^2\right)}} \qquad V_{tt1} = \frac{1 + \cos(\theta_1 - \omega)}{\sqrt{p_2}} \tag{45c}$$

Again, it is possible to write $\Delta V_2 = \sqrt{[(V_{rt1} - V_{r1})^2 + (V_{tt1} - V_{t1})^2]}$, which completes the expressions required to calculate the total $\Delta V(\Delta V_1 + \Delta V_2)$.

The next step is to obtain the analytical derivatives of the ΔV with respect to the three variables (θ_0, θ_1, and ω), set them equal to zero, and solve the resulting system of three equations and three unknowns. The solution of this system gives the transfer that extremize the ΔV. The equations are not shown here to save space, but they are efficient and easy to code.

With those equations, some examples can be solved to find the solutions for the two- and three-impulse transfers, using the theories developed in this paper. Table 1 shows the results. It includes the initial data e, e_2, a_2, and ω_2 that specify the initial and the final orbits; the solution of the three-impulse transfer developed in this paper (θ_0, θ_1, and ω); the magnitude of the impulses required by each transfer; and the savings obtained with the application of the third impulse at infinity. Note that a positive saving means that the three-impulse maneuver requires a smaller ΔV and a negative saving means that the two-impulse maneuver requires a smaller ΔV.

From Table 1 it is possible to see that positive savings are reached for many of the cases tested. However, this maneuver cannot be used in practical applications in the way it is, because of the infinity time required to complete the transfer. The importance of this maneuver is that it represents the limit case of a practical maneuver that has the second impulse applied at a finite distance (as large as the time constraints allow) with a finite (but small) ΔV. It is also possible to see that, in all of the cases studied, the change in the argument of periapse has a larger component in the second impulse ($\omega < \omega_2/2$).

Four-Impulse Transfer Through Infinity

For the most general case of two noncoaxial ellipses, it is possible to find a scheme that can decrease the total ΔV by using a fourth impulse at infinity (this idea was already known by Edelbaum[12]). This four-impulse maneuver is performed in the following four steps.

1) The first impulse is applied when the spacecraft is passing by the periapse of the initial orbit. The magnitude should be the one required to make the spacecraft achieve parabolic escape velocity

Table 2 Results for the two-, three-, and four-impulse transfers

Orbital elements				Four-impulse transfer			Four vs two	Four vs three
e	e_2	a_2	ω_2	ΔV_1	ΔV_2	ΔV_{tot}	ΔV_{SAV}	ΔV_{SAV}
0.3	0.3	10	60	0.3275	0.1036	0.4311	0.0175	0.0169
0.3	0.3	10	120	0.3275	0.1036	0.4311	0.0631	0.0450
0.3	0.3	30	60	0.3275	0.0598	0.3873	0.0467	0.0104
0.3	0.3	30	120	0.3275	0.0598	0.3873	0.0850	0.0264
0.3	0.6	10	60	0.3275	0.0747	0.4022	0.0147	0.0410
0.3	0.6	10	120	0.3275	0.0747	0.4022	0.0878	0.1009
0.3	0.6	30	60	0.3275	0.0431	0.3706	0.0461	0.0279
0.3	0.6	30	120	0.3275	0.0431	0.3706	0.1249	0.0595
0.6	0.3	10	60	0.2361	0.1036	0.3397	0.0586	0.0189
0.6	0.3	10	120	0.2361	0.1036	0.3397	0.1159	0.0461
0.6	0.3	30	60	0.2361	0.0598	0.2959	0.0624	0.0111
0.6	0.3	30	120	0.2361	0.0598	0.2959	0.1052	0.0267
0.6	0.6	10	60	0.2361	0.0747	0.3108	0.0767	0.0537
0.6	0.6	10	120	0.2361	0.0747	0.3108	0.1955	0.1041
0.6	0.6	30	60	0.2361	0.0431	0.2792	0.0977	0.0317
0.6	0.6	30	120	0.2361	0.0431	0.2792	0.1696	0.0603

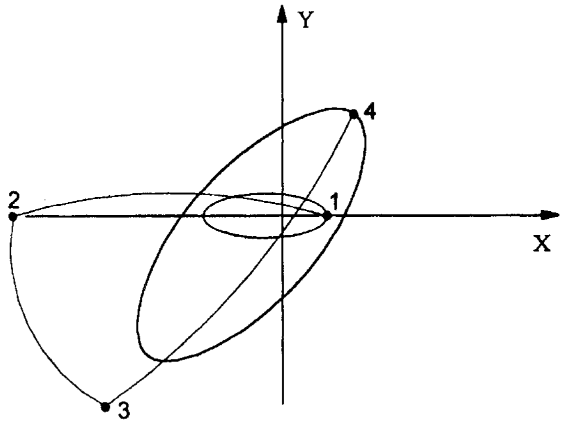

Fig. 9 Geometry of the transfer for a four-impulsive maneuver.

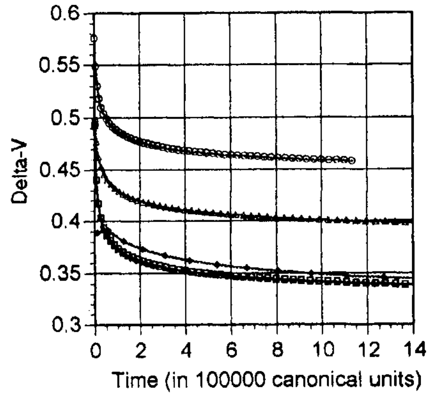

Fig. 10 ΔV vs time for practical transfers with four impulses.

at that point. Assuming a transfer from an orbit with Keplerian elements $a = 1$, $e = e$, and $\omega = 0$, the result is

$$\Delta V_1 = \sqrt{\frac{2}{(1-e)}} - \sqrt{\frac{2}{(1-e)} - 1} \tag{46}$$

2) After this first impulse, it is necessary to wait until the spacecraft reaches the apoapse of this first quasiparabolic transfer. Then a second impulse is applied with zero magnitude to circularize the orbit (remember that the radius of this circular orbit is infinite).

3) After this second impulse it is necessary to wait until the spacecraft increases its true anomaly by ω_2. At this point the spacecraft is 180 deg apart from the periapse of its final orbit. This is the right point to apply a third impulse with zero magnitude, to transfer the satellite from this circular orbit to a quasiparabolic orbit with periapse coincident with the periapse of the final orbit.

4) For the final step, it is necessary to wait until the spacecraft reaches the periapse of its second quasiparabolic transfer orbit. At this moment the fourth (final) impulse is applied to capture the spacecraft into its final orbit. Assuming a final orbit with orbital elements a_2, e_2, and ω_2, the result is

$$\Delta V_4 = \sqrt{\frac{2}{a_2(1-e_2)}} - \sqrt{\frac{2}{a_2(1-e_2)} - \frac{1}{a_2}} \tag{47}$$

Figure 9 shows a sketch of this transfer. Table 2 shows the various ΔV for the same cases studied before, as well as the savings over the two and three impulses maneuvers.

Figure 10 shows ΔV vs time for four examples of transfer: T1, $e = 0.6$, $a_2 = 10$, $e_2 = 0.6$, and $\omega_2 = 120$; T2, $e = 0.3$, $a_2 = 10$, $e_2 = 0.3$, and $\omega_2 = 60$; T3, $e = 0.3$, $a_2 = 30$, $e_2 = 0.6$, and $\omega_2 = 120$; and T4, $e = 0.6$, $a_2 = 30$, $e_2 = 0.3$, and $\omega_2 = 60$. The various ΔV go to a limit of 0.3108, 0.4311, 0.3706, and 0.2959, respectively, when the time goes to infinity.

Conclusions

A new set of equations to solve the problem of optimal transfers between two coplanar noncoaxial elliptic orbits, in a nonlinear system of three equations and three unknowns, is derived. A new approach to solve the problem of optimal transfers between two coplanar elliptic orbits by using three impulses, with the second one applied at an infinite distance, is also derived. It is an extension of the well-known bi-elliptic transfer, to be applied in the cases where the two orbits are elliptic and not coaxial. This problem is also reduced to the problem of solving a nonlinear system of three equations and three unknowns. The results show the best positions to apply both finite impulses and how to split the change in the argument of periapse between the two impulses. In all of the cases studied, the larger part of the change in the argument of periapse is done with the final impulse. The two schemes are compared by solving several transfers in both cases. A scheme to calculate minimum transfers with four impulses, two of them applied at an infinite distance, is also implemented for comparison. It proves to be useful in reducing the various ΔV in most of the cases. The regions where the use of more than two impulses can give savings in ΔV are shown for the more trivial cases of transfers (between circular orbits, circular-to-elliptic orbits, and two coaxial elliptic orbits).

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